**Problem 12.** Let $X$ and $Y$ be Banach spaces and $T \in B(X,Y)$.

(a) Show that the mapping

$$\begin{aligned} \hat{T} : X/_{\ker T} &\to \operatorname{ran} T \\ [x] &\mapsto Tx \end{aligned}$$

is well-defined, bijective and bounded with $\|\hat{T}\| = \|T\|$.

(b) Show that the following are equivalent:
   (i) $\operatorname{ran} T$ is closed,
   (ii) $\hat{T}$ has a bounded inverse,
   (iii) There exists a constant $K > 0$, so that for all $y \in \operatorname{ran} T$ there exists a $x \in X$ with $\|x\| \leq K \|y\|$ and $Tx = y$.

**Problem 13.** Let $X$ be a Banach space and $T \in B(X)$. For $\lambda \in \mathbb{C} \setminus \sigma(T)$ define $\delta(\lambda) = d(\lambda, \sigma(T)) = \inf_{\xi \in \sigma(T)} |\lambda - \xi|$. Show that $\|R_T(\lambda)\| \geq \delta(\lambda)^{-1}$.

**Problem 14.** Let $\mathcal{H}$ be a Hilbert space. An operator $T \in B(\mathcal{H})$ is called *bounded from below* if there exists a constant $C > 0$ so that for all $x \in \mathcal{H}$

$$\|Tx\| \geq C \|x\| .$$

We denote

$$\begin{aligned} \sigma_{ap}(T) &= \{\lambda \in \mathbb{C} \mid \lambda I - T \text{ is not bounded from below}\}, \\ \sigma_d(T) &= \{\lambda \in \mathbb{C} \mid \operatorname{ran}(\lambda I - T) \text{ is not closed}\}. \end{aligned}$$

Show:

(a) $\sigma_{ap}(T) = \{\lambda \mid \exists (x_n) \subseteq \mathcal{H} \text{ with } \|x_n\| = 1 \text{ and } \|(\lambda I - T)x_n\| \xrightarrow[n\to\infty]{} 0\}$
(b) $\sigma_{ap}(T)$ is closed.
(c) $\sigma(T) = \sigma_{ap}(T) \cup \sigma_d(T)$.
(d) The boundary of $\sigma(T)$ is contained in $\sigma_{ap}(T)$.
(e) $\sigma_p(T^*) = \overline{\sigma_d(T)}$.
(f) If $T$ is normal then $\sigma_d(T) \setminus \sigma_{ap}(T) = \emptyset$, i.e. $\sigma(T) = \sigma_{ap}(T)$.

**Problem 15.** Let $X$ be a Banach space and $S, T \in B(X)$.

(a) Show that the following do not hold in general:

$$(\dagger) \qquad r(ST) \leq r(S)\, r(T) \qquad r(S+T) \leq r(S) + r(T)$$

(b) Show that both inequalities in (†) hold if $S$ commutes with $T$ (i.e. $ST = TS$).

**Problem 16.** Let $X$ be a Banach space and $T_n \in B(X)$ a sequence of invertible operators. Assume that the sequence has a limit $T = \lim T_n$, which commutes with all $T_n$ (i.e. $TT_n = T_nT$ for all $n$).

(a) Show that in general $T$ is not necessarily invertible.
(b) Show that if we also assume that $\sup_n r(T_n^{-1}) < \infty$, then $T$ is invertible.